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Scattering of Solitons in Derivative Nonlinear Schrödinger Model

Hyunsoo Min

Department of Physics
Seoul City University
Seoul, 130-743, Korea

and

Q-Han Park[1]

Department of Physics
and
Research Institute of Basic Sciences
Kyunghee University
Seoul, 130-701, Korea

ABSTRACT

We show that the chiral soliton model recently introduced by Aglietti et al. can be made integrable by adding an attractive potential with a fixed coefficient. The modified model is equivalent to the derivative nonlinear Schrödinger model which does not possess parity and Galilean invariance. We obtain explicit one and two classical soliton solutions and show that in the weak coupling limit, they correctly reproduce the bound state energy as well as the time delay of two-body quantum mechanics of the model.

[1] E-mail address; qpark@nms.kyunghee.ac.kr

It has been known for sometime that soliton solutions to certain nonlinear field equations can be associated with elementary particles in quantum field theory. In particular, the nonlinear Schrödinger system shows that its classical solitons behave as quantized particles of the same theory. This may be compared with the sine-Gordon solitons which are associated with quantized elementary particles of the massive Thirring model. Recently, this correspondence has been extended to a new type of 1+1-dim nonlinear field equation[1, 2], which arose in the study of the dimensionally reduced 2+1-dim nonlinear Schrödinger model coupled to Chern-Simons gauge theory[3], breaking the Galilean invariance. It has been noted that this theory supports a chiral soliton solution, whose mass formula justifies the interpretation of a soliton as a bound state of elementary particles of the quantized theory in the weak coupling limit[1]. However, one serious drawback of the theory was its lacking of integrability structure which made impossible of finding multi-soliton solutions and their subsequent scattering behaviors. In this Letter, we show that this problem can be cured nicely when we add an attractive potential term to the theory with a fixed coefficient. This allows us to find exact one and two soliton solutions and address their quantum mechnical particle-like behavior. We find that one soliton solution reproduces the bound state energy of two identical particles, and reduces to the one soliton solution of Ref.[1] in the weak coupling limit. Two soliton solution describes the soliton-soliton scattering from which we obtain the time delay of each solitons during the scattering process. It is shown that these time delays agree with those of quantum particle scattering thereby confirming the quantum particle interpretation of solitons of the present model.

The model we consider is given by the Lagrangian,

$$L = \int dx \Big[i\hbar \Psi^* \partial_t \Psi - \frac{\hbar^2}{2m} |(\partial_x \mp i\kappa^2 \rho)\Psi|^2 + b\frac{\hbar^2\kappa^4}{2m}\rho^3 \Big] \tag{1}$$

where $\rho = |\Psi|^2$, κ^2 is the coupling constant and b is a number to be fixed later. Note that this Lagrangian is not invariant under parity but becomes invariant if we combine parity with changing the sign of the coupling constant,

$$x \to -x, \quad \kappa^2 \to -\kappa^2. \tag{2}$$

Thus from now on, we will restrict only to the upper sign case. The lower sign case can be obtained directly by changing x to $-x$ in the upper sign case. The equation of motion is

$$\begin{aligned} i\hbar\partial_t \Psi &= -\frac{\hbar^2}{2m}(\partial_x - i\kappa^2\rho)^2\Psi - \hbar\kappa^2 j \Psi - 3b\frac{\hbar^2\kappa^4}{2m}\rho^2\Psi \\ &= -\frac{\hbar^2}{2m}\partial_x^2\Psi + 2i\frac{\hbar^2\kappa^2}{m}\rho\partial_x\Psi - \frac{3}{2}(b-1)\frac{\hbar^2\kappa^4}{m}\rho^2\Psi. \end{aligned} \tag{3}$$

The current j, defined by

$$j(x) = \frac{\hbar}{2im}\Big[\Psi^*(\partial_x - i\kappa^2\rho)\Psi - \Psi(\partial_x + i\kappa^2\rho)\Psi^*\Big], \tag{4}$$

satisfies the continuity equation

$$\partial_t \rho + \partial_x j = 0 \tag{5}$$

when the equation of motion (3) is used. The $b = 0$ case, i.e. without the potential term in the Lagrangian (1), was considered in Ref.[1] and one soliton solution with chiral behavior has been obtained. However, the Painlevé test shows that Eq.(3) becomes integrable only for the value $b = 1$ [4], in which case the equation is known as the derivative nonlinear Schrödinger equation of type II. Henceforth, we will set $b = 1$. In order to compare with other literatures concerning about the derivative nonlinear Schrödinger equation, we may take a transform of Ψ with an arbitrary constant a,

$$\Psi = e^{-ia\kappa^2 \int^x dy\rho(y)} \psi, \tag{6}$$

which brings Eq.(3) into a different, yet equivalent expression,

$$i\hbar\partial_t \psi = -\frac{\hbar^2}{2m}\partial_x^2 \psi + i\kappa^2 \frac{\hbar^2}{m}[(2+a)\rho\partial_x \psi + a\psi^2 \partial_x \psi^*] + \frac{\hbar^2\kappa^4}{2m} a(2-a)\rho^2 \psi, \tag{7}$$

where $\rho = |\Psi|^2 = |\psi|^2$. For $a = -1$, we recover the expression of $j\psi$-coupling given in Ref.[1] with an additional term,

$$i\hbar\partial_t \psi = -\frac{\hbar^2}{2m}\partial_x^2 \psi - 2\hbar\kappa^2 j\psi - \frac{3\hbar^2\kappa^4}{2m}\rho^2 \psi. \tag{8}$$

For $a = 2$, Eq.(7) reduces to the derivative nonlinear Schrödinger equation of type I;

$$i\hbar\partial_t \psi = -\frac{\hbar^2}{2m}\partial_x^2 \psi + 2i\frac{\hbar^2\kappa^2}{m}\partial_x(\rho\psi). \tag{9}$$

For the sake of brevity, we introduce dimensionless coordinates

$$X = 4\kappa^2 x \quad , \quad T = 8(\hbar\kappa^4/m)t \tag{10}$$

and express Eq.(3) with $b = 1$ in a dimensionless form,

$$i\partial_T \Psi = -\partial_X^2 \Psi + i\rho\partial_X \Psi. \tag{11}$$

A general scheme for solving the derivative nonlinear Schrödinger equation has been given by Kaup and Newell in terms of the inverse scattering method[5]. Following their method, we construct explicitly one and two soliton solutions of Eq.(11) in the following. The one soliton solution is given by

$$\Psi = \sqrt{\rho}\exp(-i\Theta + \frac{i}{4}\int^X \rho), \tag{12}$$

where

$$\rho = \frac{2\mu^2|V|}{\sqrt{1+\mu^2}\cosh(\mu V[X-VT]) + \mathrm{Sign}(V)}$$
$$\Theta = -\frac{1}{2}V[X-VT] - \frac{1}{4}(1+\mu^2)V^2T + \theta. \tag{13}$$

V is the velocity of the soliton and θ is an arbitrary initial phase. The parameter $\mu \geq 0$ accounts for the mass of the soliton as shown below. For finite μ, note that the soliton solution Ψ vanishes when the velocity V is set to zero. This means that there is no static finite mass soliton (with time periodic phase) thus one can not simply obtain the moving one soliton solution by boosting the static one as in the case of the usual nonlinear Schrödinger equation. This clearly shows that the system is lacking of Galilean invariance. Nevertheless, it is intriguing to see that, for infinite μ such that $\mu V = \alpha$ is finite, Eq.(12) admits a static soliton,

$$\Psi e^{-\frac{i}{4}\int^X \rho} = \Big[\frac{2\alpha}{\cosh(\alpha X)}\Big]^{\frac{1}{2}} \exp(i\alpha^2 T/4). \tag{14}$$

For later use, we present an equivalent expression of Eq.(12) which is useful in comparison with the two soliton scattering process,

$$\Psi e^{-\frac{i}{2}\int^X \rho} = \frac{\mu|V|}{\sqrt{|\lambda|}} \frac{\exp(\Delta)\exp(-i\Theta)}{\exp(2\Delta) - \lambda/|\lambda|} \tag{15}$$

where

$$\Delta = \frac{1}{2}\mu|V|[X-VT], \quad \lambda = \frac{1}{4}(-V + i\mu|V|). \tag{16}$$

The system possesses infinitely many conserved charges among which the first few are playing the role of dynamical parameters in the context of particle interpretation. In terms of dimensionful parameters, we evaluate those quantities explicitly on the one soliton solution with the following result; the constituent number N is

$$N = \int_{-\infty}^{\infty} dx\rho = \frac{1}{\kappa^2}\tan^{-1}(\mu v/|v|); \quad 0 < \kappa^2 N < \pi \tag{17}$$

while the momentum P is

$$P = \int_{-\infty}^{\infty} dx \frac{\hbar}{2i}\Big[\Psi^*\partial_x\Psi - \partial_x\Psi^*\Psi\Big]$$
$$= (\frac{\mu m}{\kappa^2})|v| \equiv Mv. \tag{18}$$

Note that the momentum P is always positive regardless of the sign of the velocity. The soliton mass M may be written in terms of the constituent number N,

$$M = \frac{\mu m}{\kappa^2}\frac{v}{|v|} = \frac{m}{\kappa^2}\tan(\kappa^2 N). \tag{19}$$

The energy E, defined by

$$E = \int_{-\infty}^{\infty} dx \left[\frac{\hbar^2}{2m} |(\partial_x - i\kappa^2 \rho)\Psi|^2 - \frac{\hbar^2 \kappa^4}{2m} \rho^3 \right], \tag{20}$$

becomes

$$E = \frac{1}{2} vP = \frac{1}{2M} P^2. \tag{21}$$

Thus, it satisfies the particle dispersion relation. For small κ, we may expand M as

$$M = m(N + \frac{1}{3}\kappa^4 N^3) + O(\kappa^6). \tag{22}$$

In the weak coupling limit where $\kappa^2 N$ is kept small, the one soliton solution in Eq.(12) agrees precisely with that of Ref.[1]. However, one marked difference is that unlike the one soliton of Ref.[1] which admits only positive velocity, our case admits both signs of velocity. Indeed, negative velocity in our context requires a strong coupling, $\pi/2 < \kappa^2 N < \pi$, thus in comparing with particles in the weak coupling limit, we take only the positive velocity case.

The quantum mechanics of the two-body system for our model is the same as that appeared in Ref.[1] since the potential term added to the Lagrangian (1) does not affect the two body system. Define the two-body wave function by

$$\Phi(x_1, x_2) = \langle 0|\Psi(x_1)\Psi(x_2)|2\rangle. \tag{23}$$

Because of time and space translational invariance, one can set

$$\Phi(x_1, x_2; t) = \exp(-iEt/\hbar)\exp(iP(x_1 + x_2)/2\hbar)u(x_1 - x_2), \tag{24}$$

so that the two-body equation for the relative motion becomes

$$(-\frac{\hbar^2}{m}\frac{d^2}{dx^2} + \frac{P^2}{4m} - \frac{P}{m}\hbar\kappa^2\delta(x))u(x) = Eu(x). \tag{25}$$

This possesses a bound state provided P is posivtive. In which case, the total energy E is

$$E = \frac{P^2}{4m}(1 - \kappa^4). \tag{26}$$

On the other hand, if we assume the same type of one-loop modification to Eq.(22) as suggested in Ref.[1, 6],

$$M_{\text{semiclassical}} = mN + \frac{1}{3} m\kappa^4 (N^3 - N), \tag{27}$$

we get for $N = 2$

$$M_{\text{semiclassical}} = 2m(1 + \kappa^4) \tag{28}$$

which is consistent with Eq.(26) for small κ.

A straightforward calculation following the inverse scattering method for the derivative nonlinear Schrödinger equation shows that the exact two soliton solution can be given by

$$\begin{aligned}\Psi e^{-\frac{i}{2}\int^X \rho} &= \Big[2e^{-\Delta_1 - i\Theta_1} + 2e^{-\Delta_2 - i\Theta_2} + A_{12}e^{-2\Delta_1-\Delta_2-i\Theta_2} + A_{21}e^{-2\Delta_2-\Delta_1-i\Theta_1}\Big] \\ &\quad \times\Big[1 + De^{-2\Delta_1-2\Delta_2} + (C_{11}e^{-2\Delta_1} + C_{12}e^{-\Delta_1-\Delta_2+i\Theta_1-i\Theta_2} + 1 \leftrightarrow 2)\Big]^{-1} \end{aligned} \tag{29}$$

where

$$\begin{aligned} A_{ij} &= 2\lambda_i\Big(\frac{1}{\lambda_i - \lambda_j^*} - \frac{1}{\lambda_i - \lambda_i^*}\Big)^2 \ , \quad C_{ij} = \frac{\lambda_i}{(\lambda_i - \lambda_j^*)^2} \\ D &= \lambda_1\lambda_2\Big(\frac{1}{|\lambda_1 - \lambda_2^*|^2} + \frac{1}{(\lambda_1 - \lambda_1^*)(\lambda_2 - \lambda_2^*)}\Big)^2 . \end{aligned} \tag{30}$$

$\Theta_1(\Theta_2)$ and $\Delta_1(\Delta_2)$ are as in Eqs.(13) and (16) with velocity $V_1(V_2)$. Justification for the above expression may follow from the following asymptotic behavior; in the limit $t \to \infty$ with Δ_1 fixed so that $\Delta_2 \to \infty$ for $V_1 > V_2$, the two soliton solution in Eq.(29) approaches to

$$\begin{aligned} \Psi e^{-\frac{i}{2}\int^X \rho} &= \frac{\mu_1\sqrt{|V_1|}}{\sqrt{|\lambda_1|}}\frac{e^{\Delta_1 - \phi_I}e^{-i\Theta_1}}{e^{2(\Delta_1-\phi_I)} - \lambda_1/|\lambda_1|} \\ \phi_I &\equiv \ln(\sqrt{1+\mu_1^2}/\mu_1^2) \end{aligned} \tag{31}$$

which is precisely the 1-soliton in Eq.(15) moving with velocity V_1. Also, another limit $t \to -\infty$ with Δ_1 fixed and $\Delta_2 \to -\infty$ results in the 1-soliton expression

$$\begin{aligned} \Psi e^{-\frac{i}{2}\int^X \rho} &= \frac{\mu_1\sqrt{|V_1|}}{\sqrt{|\lambda_1|}}\frac{e^{\Delta_1 - \phi_F}e^{-i\Theta_1+2i\phi_P}}{e^{2(\Delta_1-\phi_F)} - \lambda_1/|\lambda_1|} \\ \phi_F &\equiv \phi_I + \ln\Big(\frac{|\lambda_1 - \lambda_2|^2}{|\lambda_1 - \lambda_2^*|^2}\Big) \\ \tan\phi_P &\equiv \frac{(V_2 - V_1)^2 + (\mu_1 V_1 + \mu_2 V_2)^2 - 2\mu_2 V_2(\mu_1 V_1 + \mu_2 V_2)}{2\mu_2 V_2 (V_2 - V_1)} . \end{aligned} \tag{32}$$

One could repeat the same limiting procedure but with Δ_2 fixed and obtain the other soliton sector moving with velocity V_2. This shows clearly that Eq.(29) is a two soliton solution describing the soliton-soliton scattering. The term ϕ_I measures the choice of time coordinate orgin $t = 0$, which can be simply removed by shifting the orgin. However, the difference $\phi_F - \phi_I$ is invariant under time translation and measures the time delay. For example, the time delay of the soliton with velocity V_1 due to the scattering is given by,

$$[\Delta T]_1 = \frac{2}{\mu_1 V_1^2}\ln\Big[1 - \frac{4\mu_1\mu_2 V_1 V_2}{(V_1 - V_2)^2 + (\mu_1 V_1 + \mu_2 V_2)^2}\Big]. \tag{33}$$

In terms of dimensionful parameters, this becomes with $\mu_1 = \mu_2$,

$$
\begin{aligned}
[\Delta t]_1 &= \frac{\hbar}{2\kappa^2 E_1} \ln \Big[1 - \frac{4\kappa^4 (M/m)^2 \sqrt{E_1 E_2}}{(\sqrt{E_1} - \sqrt{E_2})^2 + \kappa^4 (M/m)^2 (\sqrt{E_1} + \sqrt{E_2})^2} \Big] \\
&= -2\hbar\kappa^2 (M/m)^2 \frac{\sqrt{E_2}}{\sqrt{E_1}} \frac{1}{(\sqrt{E_1} - \sqrt{E_2})^2} + \mathcal{O}(\kappa^4)
\end{aligned}
\tag{34}
$$

where we have expressed velocities in terms of energies.

In the two-body quantum mechanics, the phase shift occuring in the scattering of two identical particles, each having positive kinetic energy E_1 and E_2, can be easily computed to give

$$
\begin{aligned}
2\delta(E_1, E_2) &= 2 \tan^{-1} \Big(\kappa^2 \frac{\sqrt{E_1} + \sqrt{E_2}}{\sqrt{E_1} - \sqrt{E_2}} \Big) \\
&= 2\kappa^2 \frac{\sqrt{E_1} + \sqrt{E_2}}{\sqrt{E_1} - \sqrt{E_2}} + \mathcal{O}(\kappa^4).
\end{aligned}
\tag{35}
$$

Using the relation between the phase shift and the time delay[7], we find the time delay for the particle with energy E_1,

$$
[\Delta t]_1 = \hbar \frac{\partial}{\partial E_1} 2\delta(E_1, E_2) = -2\hbar\kappa^2 \frac{\sqrt{E_2}}{\sqrt{E_1} (\sqrt{E_1} - \sqrt{E_2})^2} + \mathcal{O}(\kappa^4).
\tag{36}
$$

In the leading order, this agrees with Eq.(34) when the soliton mass is taken to be that of a quantum particle so that $N = 1$, and the velocities are all taken positive values. This strengthens the quantum particle interpretation of classical solitons for the present case. In the case of the usual nonlinear Schrödinger equation, same result has been obtained in [8].

In conclusion, we have shown that the potential term added to the theory of Ref.[1] as in Eq.(1) makes the theory integrable and the one and two soliton solutions correctly reproduce the characteristics of two-body quantum mechanics in the weak coupling limit. It was also shown that the theory possesses soliton solutions with negative velocity which do not have a direct particle interpretation. Since it requires strong coupling $\pi/2 < \kappa^2 N < \pi$, this seems to suggest a collective motion of a bound state of particles with large N which however stays as an open problem.

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